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Admissible Subspaces and the Denseness of the Intersection of the Domains of Semigroup Generators

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Let $T_i = \{T_i(t) : t \geqslant 0\}$ be a (C_0) semigroup of linear operators on a Banach space X, with infinitesimal generator A_i, $i = 1, 2$. If Y is a dense Banachable subspace of X left invariant by T_1, then it is shown that in many cases $D(A_1) \cap Y$ is dense in X, where $D(A_1)$ is the domain of A_1. In particular, if X is either separable or reflexive and if T_1 leaves $D(A_2)$ invariant, then $D(A_1) \cap D(A_2)$ is dense in X.

1. Introduction

Let $T = \{T(t) : t \geqslant 0\}$ be a (C_0) semigroup of linear operators on a Banach space X, and let Y be a dense Banachable subspace of X left invariant by T. While in general it is not true that the restriction of T to Y is strongly continuous when Y is given its Banach space structure, we shall show that in many cases this restriction is strongly continuous for $t > 0$ and the domain of the infinitesimal generator of T intersects Y in a dense subspace of X. As a consequence we obtain a sufficient condition for the intersection of the domains of two (C_0) semigroup generators to be dense.

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2. Types of Subspaces

Let X be a Banach space. A subspace Y of X is *Banachable* if there is a norm which makes Y into a Banach space (of course, this is simply a condition on the cardinality of Y as an abstract vector space). $[Y]$ will denote Y equipped with its Banach space structure. We shall call Y a *type* I *subspace* of X if Y is a dense Banachable subspace and the natural injection from $[Y]$ to X is continuous. A *type* II *subspace* of X is a type I subspace Y such that $[Y]$ is homeomorphic to X. A *type* III *subspace* of X is a type I subspace Y such that $[Y]$ is linearly homeomorphic to X. A *type* IV *subspace* of X is the domain of a closed densely defined linear operator on X having nonempty resolvent set.

Remarks. 1. *A type* J *subspace is a type* J–1 *subspace,* J = II, III, IV. This is immediate for J = II or III. For J = IV, let B be a closed operator having Y as its domain and 0 in its resolvent set. Y is a Banach space under the graph norm of B, and $B : [Y] \to X$ is the required linear homeomorphism.

2. *A subspace Y is type* III *if and only if it is type* IV. We have just seen that type IV subspaces are type III. Conversely, if $Y \subset X$ is type III, let $B : [Y] \to X$ be a linear homeomorphism. We regard B as an operator on X with domain Y. Then $B^{-1} : X \to Y$ is bounded. Therefore B is closed and 0 is in its resolvent set.

3. *A type* II *subspace need not be a type* III *subspace.* For example let $X = L^2(0, 1)$, $Y = L^p(0, 1)$, $2 < p < \infty$. Obviously the injection from $[Y]$ to X is continuous, and $[Y]$ is homeomorphic to X (indeed, any two separable Banach spaces are homeomorphic), but Y is not a type III subspace of X.

4. *A type* I *subspace need not be a type* II *subspace.* For example, let $X = L^2(0, 1)$ and $Y = L^\infty(0, 1)$. The point is that here X is separable but $[Y]$ is not. A related example is used in Proposition 1 below. (On the other hand, if X is separable and Y is a type I subspace with $[Y]$ reflexive, it is not difficult to see that $[Y]$ is separable and therefore type II.[1])

[1] Here is a proof. Let $i : [Y] \to X$ be the inclusion map, let $\{x_n\}$ be a dense sequence in the unit sphere of X, and choose $\varphi_n \in X^*$ with $\| \varphi_n \| = \varphi_n(x_n) = 1$. Then $\{\varphi_n\}$ separates points in X, thus $\{\varphi_n \circ i\} \subset [Y]^*$ separates points in Y. Hence the norm-closed linear span S of $\{\varphi_n \circ i\}$ is a separable weak* dense subspace of $[Y]^*$; consequently $S = [Y]^*$ since $[Y]$ is reflexive. Thus $[Y]^*$ is separable and so is $[Y]$.

3. Admissible Subspaces for Semigroups

Let $T = \{T(t) : t \geqslant 0\}$ be a semigroup of bounded operators on a Banach space X which is strongly continuous for $t > 0$, i.e., for each $x \in X$ the map $t \to T(t)x$ is continuous on $(0, \infty)$. In the terminology of Hille–Phillips [3], T is a (C_0) semigroup if in addition $T(t)x \to x$ as $t \to 0$ for each $x \in X$. Whether or not T is (C_0), the *infinitesimal operator* A_0 of T is defined by

$$A_0 x = \lim_{t \to 0} t^{-1}[T_t x - x]$$

on the domain for which the limit exists. In case T is a (C_0) semigroup, A_0 is closed and densely defined, and is called the *infinitesimal generator* of T. (We shall not make use of the notion of infinitesimal generator for more general classes of semigroups; cf. [3, p. 344].)

According to [3, p. 307] the domain of A_0 and the union of the ranges of the operators $T(t)$ for $t > 0$ have the same closure in X. We shall say that T is an *essential* semigroup provided this closure is the whole of X. Thus any (C_0) semigroup is essential.

Now let T be an essential semigroup on X and let Y be a type I subspace of X. We say that Y is *weakly T-admissible* if $T(t)Y \subset Y$ for each $t > 0$ and if the restriction of T to $[Y]$ is strongly continuous for $t > 0$ and essential. We shall say that Y is *T-admissible* provided $T \mid [Y]$ is a (C_0) semigroup on Y. We make analogous definitions for one-parameter *groups* of operators; of course for groups there is no difference between admissibility and weak admissibility. The notion of admissibility is useful in several contexts; cf. [2, 4].

Proposition 1. *There exists a Banach space X, a type* I *subspace Y of X, and a strongly continuous one-parameter group $T = \{T(t) : -\infty < t < \infty\}$ on X which leaves Y invariant, but for which Y is not (weakly) admissible.*

Proof. For $1 \leqslant p < \infty$ let $X = L_p(-\infty, \infty)$. Define $T(t)f(x) = f(x + t)$ for $f \in X$. Then T is a strongly continuous one-parameter group. Let $Y = \{f \in X : f \text{ is bounded and continuous}\}$. The norm on Y defined by

$$|f| = \|f\|_p + \|f\|_\infty$$

makes Y into a type I subspace of X which is obviously left invariant by each $T(t)$. Moreover each $T(t)$ is an isometry on $[Y]$. To see that Y is not T-admissible, let f be any function in Y which is not uniformly continuous on $(-\infty, \infty)$. Then for some $\epsilon > 0$ and all $t \neq 0$ we have $|T(t)f - f| \geqslant \|T(t)f - f\|_\infty \geqslant \epsilon > 0$. So strong continuity fails for T on $[Y]$, and hence weak admissibility fails for T on $[Y]$. $\square$

The next result shows that the distinction between admissibility and weak admissibility is a real one even for (C_0) semigroups T on X.

PROPOSITION 2. *There exists a separable Hilbert space X, a type* III *subspace Y of X, and a (C_0) semigroup T on X for which Y is weakly admissible but not admissible.*

Proof. We can adapt for this purpose a construction of Hille–Phillips [3, p. 371]. Let X be the space of square-summable sequence pairs $\{(x_n, y_n)\}$ with $\|\{(x_n, y_n)\}\|^2 = \sum_{n=1}^{\infty} (|x_n|^2 + |y_n|^2)$. Let Y be the subspace consisting of all $\{(x_n, y_n)\}$ such that

$$\sum_{n=1}^{\infty} (|x_n|^2 + n|y_n|^2) < \infty.$$

Then Y is a type III subspace of X with $[Y]$ a Hilbert space.

For $t \geqslant 0$ define $T(t)\{(x_n, y_n)\} = \{(x_n', y_n')\}$, where

$$x_n' = e^{-nt}(x_n \cos nt - y_n \sin nt),$$

$$y_n' = e^{-nt}(x_n \sin nt + y_n \cos nt).$$

It is clear that T is a (C_0) contraction semigroup on X. Moreover it is not hard to see that $T(t)Y \subset Y$ for $t > 0$ and that on $[Y]$

$$|T(t)| \leqslant \sup_{n \geqslant 1}[n^{1/2}e^{-nt}] \leqslant (2et)^{-1/2}.$$

Thus, $T(t)$ defines a bounded operator on $[Y]$, indeed uniformly bounded for $t \geqslant \delta > 0$. It is very easy to check that $T \mid [Y]$ is strongly continuous for $t > 0$. Moreover, the domain of its infinitesimal operator contains all sequences with a finite number of nonzero terms, and is therefore dense in $[Y]$, proving that $T \mid [Y]$ is an essential semigroup.

However, if $v_k = \{(\delta_{nk}, 0)\}$ and we take $t_k = \pi/2k$ so that $\sin(kt_k) = 1$, $\cos(kt_k) = 0$, we have $T(t_k)v_k = \{(0, \delta_{nk})\}$, whence

$$|T(t_k)v_k| = e^{-\pi/2}\sqrt{k}\,|v_k|.$$

This shows that $|T(t)|$ is unbounded as $t \to 0$, so $T \mid [Y]$ is not of class (C_0). $\square$

4. Main Results

We shall show now that examples of the type of Proposition 1 are in a sense "pathological," and that under very general hypotheses T-invariant subspaces are weakly T-admissible. The main tool we use is the basic theory of Borel spaces, for which a convenient reference is the book of Parthasarathy [5, Chap. 1].

THEOREM 3. *Let T be an essential semigroup on a separable Banach space X, strongly continuous for $t > 0$. Let Y be a type* I *subspace of X left invariant by T with $[Y]$ separable. Let $Z \subset X$ be the closure in $[Y]$ of $\bigcup \{T(t)Y : t > 0\}$. Then Z is a weakly T-admissible type* I *subspace of X.*

Proof. Being complete and separable, X and $[Y]$ are standard Borel spaces. The injection $i : [Y] \to X$ is continuous, and therefore by [5, p. 21] i is a Borel isomorphism onto its range $Y \subset X$. For $y \in Y \subset X$, $t \to T(t)y$ is continuous and therefore Borel measurable from $(0, \infty)$ to $Y \subset X$, whence $i^{-1}(T(t)y)$ is Borel measurable from $(0, \infty)$ to $[Y]$. Also each $T(t)$ is a bounded operator on $[Y]$ by the closed graph theorem. Therefore, $T \mid [Y]$ is a strongly measurable semigroup on Z as defined above, and the infinitesimal operator of $T \mid [Y]$ has domain dense in Z. So $T \mid [Z]$ is an essential semigroup strongly continuous for $t > 0$ (cf. [3, pp. 305–307]).

It remains to show that Z is dense in X. Given $x \in X$ we can find a sequence $t_n > 0$ and vectors $y_n \in X$ with $T(t_n)y_n \to x$, because by assumption T is essential. Since each $T(t_n)$ is bounded and Y is dense in X, we can take the above y_n to be in Y. But then $z_n = T(t_n)y_n \in Z$, so $x \in \bar{Z}$. $\square$

Remarks. 1. For a simple example where Z is a proper subspace of Y, take $X = L^2(0, 1)$ and $Y = C[0, 1]$ with $T(t)f(x) = x^t f(x)$. Here $Z = C_0[0, 1] = \{f \in Y : f(0) = 0\}$.

Another example is given by $X = L^2(0, 1)$,

$$Y = H^1(0, 1) = \{f \in L^2 : f \text{ is absolutely continuous and } f' \in L^2\}$$

with $| f |^2 = \| f \|_2^2 + \| f' \|_2^2$. Here we take $T(t)f(x) = e^{-t/x}f(x)$. It is not hard to verify that $T(t)Y \subset Y$ and $T(t)$ is strongly continuous for $t > 0$ on $[Y]$. Here $Z = H_0^1(0, 1) = \{f \in Y : f(0) = 0\}$.

2. In the first of the above examples Y was not reflexive. In the second it is not hard to see that $T(t) \mid [Y]$ is not uniformly bounded as $t \to 0$. In fact if neither of these undesirable things happens, and

if T is a (C_0) semigroup on X, we can deduce that $Y = Z$ and that Y is admissible. This follows from the following result.[2]

LEMMA 4. *Let Y be a reflexive Banach space. Let $T(t)$ be a semigroup on Y, strongly continuous for $t > 0$, and such that $T(t)y = 0$ for all $t > 0$ implies that $y = 0$. (Note that the latter condition holds if $T(t)$ is the restriction to Y of a (C_0) semigroup on a bigger space X.) Assume in addition that $|T(t)|$ is uniformly bounded for t near to 0. Then T is a (C_0) semigroup on Y.*

Proof. Consider the adjoint semigroup $T(t)^*$ on Y^*. The norm $|T(t)^*| = |T(t)|$ is bounded near 0. Also, $\bigcup \{T(t)^*Y^* : t > 0\}$ is dense in Y^*. Otherwise, since Y is reflexive, there would be a nonzero $y \in Y$ with

$$(T(t)^*y^*)(y) = 0 \qquad \text{for all} \quad t > 0, \quad y^* \in Y^*;$$

that is, $y^*(T(t)y) = 0$ so that $T(t)y = 0$ for all $t > 0$. But by assumption this implies $y = 0$.

Note that for all $y^* \in Y^*$, $t \to T(t)^*y^*$ is weakly continuous for $t > 0$. By [3, p. 306] this implies strong continuity for $t > 0$. The uniform boundedness of $|T(t)^*|$ near 0 together with the density of the ranges just proved implies readily that $T(t)^*$ is (C_0). But then a second dualization implies that $\bigcup \{T(t)Y : t > 0\}$ is dense in Y, so that T is (C_0) on Y. □

PROPOSITION 5. *Let T be an essential semigroup on X, strongly continuous for $t > 0$, with infinitesimal operator A. Let Y be a weakly T-admissible type* I *subspace of X. Then $\mathscr{D}^\infty(A) \cap Y$ is dense in X, where $\mathscr{D}^\infty(A) = \bigcap_{n=1} \mathscr{D}(A^n)$ and $\mathscr{D}(A^n)$ is the domain of A^n.*

Proof. Let A_Y denote the infinitesimal operator of the semigroup $T \mid [Y]$. Then $A_Y \subset A$ since the injection from $[Y]$ to X is continuous. By [3, p. 308] $\mathscr{D}^\infty(A_Y)$ is dense in Y and *a fortiori* in X. Therefore $\mathscr{D}^\infty(A) \cap Y \supset \mathscr{D}^\infty(A_Y)$ is dense in X. □

We now come to our main results.

THEOREM 6. *Let X be a Banach space, and let T be an essential semigroup on X, strongly continuous for $t > 0$, with infinitesimal*

[2] T. Kato has noted that an alternate proof of Lemma 4 can be based on [T. Kato, Remarks on pseudo-resolvents and infinitesimal generators of semi-groups, *Proc. Japan Acad.* **35** (1959), 467–468].

operator A. Let Y be a type I *subspace of X left invariant by T. Assume either*

(i) *X and $[Y]$ are separable*

or

(ii) *$[Y]$ is reflexive.*

Then $\mathscr{D}^\infty(A) \cap Y$ is dense in X.

An immediate consequence of this is the following:

THEOREM 7. *Let A_1, A_2 be the infinitesimal generators of (C_0) semigroups T_1, T_2 on a Banach space X which is either separable or reflexive. If T_1 leaves $\mathscr{D}(A_2)$ invariant, then $\mathscr{D}^\infty(A_1) \cap \mathscr{D}(A_2)$ is dense in X.*

We simply apply Theorem 6, taking $[Y]$ to be $\mathscr{D}(A_2)$ with the graph norm; the resolvent of A_2 gives an isomorphism from X onto $[Y]$.

The density of $\mathscr{D}(A_1) \cap \mathscr{D}(A_2)$ is a condition that arises in the applications, cf., for instance, DaPrato [1].

Proof of Theorem 6. We first do case (i). By Theorem 3, X has a type I subspace Z $(\subset Y)$ which is weakly admissible for T. Hence, by Proposition 5, $\mathscr{D}^\infty(A) \cap Y \supset \mathscr{D}^\infty(A) \cap Z$ is dense in X.

For the proof under assumption (ii) we shall require two lemmas. We will use the following notation: $\|\cdot\|$ is the norm in X, $|\cdot|$ the norm in $[Y]$, B is the unit ball in $[Y]$, and B^* is the unit ball in $[Y]^*$.

LEMMA 8. *Let X be a Banach space and Y a type* I *subspace of X. Let T be a semigroup on X, strongly continuous for $t > 0$, which leaves Y invariant. Then $|T(t)|$ is a locally bounded function on* $(0, \infty)$.

Proof. Let $i : [Y] \to X$ be the natural inclusion map. Then $i^* : X^* \to [Y]^*$ is a continuous embedding of X^* as a weak* dense subspace of $[Y]^*$. By a well-known result of Banach [3, p. 39], $i^*(X^*) \cap B^*$ is weak* dense in B^*.

We have

$$\begin{aligned} |T(t)| &= \sup\{|T(t)y| : y \in B\} \\ &= \sup\{|\langle \varphi, T(t)y\rangle| : \varphi \in B^*, y \in B\} \end{aligned}$$

and therefore, by the weak* density of $i^*(X^*) \cap B^*$ in B^*,

$$|T(t)| = \sup\{|\langle i * \psi, T(t)y\rangle| : y \in B, \psi \in (i^*)^{-1}B^*\}.$$

But for $\psi \in X^*$, $\langle i^*\psi, T(t)y\rangle = \langle \psi, T(t)y\rangle$ is continuous on $(0, \infty)$. Hence $t \to | T(t)|$, being a supremum of continuous functions, is lower semicontinuous and thus measurable. It is, of course, submultiplicative as well, and therefore bounded on compact subsets of $(0, \infty)$ by [3, p. 241]. □

LEMMA 9. *Let the hypotheses of Lemma 8 hold, and in addition assume that* $[Y]$ *is reflexive. Then for each* $y \in Y$, *the set* $\{T(t)y : t > 0\}$ *spans a separable subspace of* $[Y]$.

Proof. Given $y \in Y$, let Y_1 be the closed subspace of $[Y]$ spanned by $\{T(t)y : t > 0,\ t \text{ rational}\}$. By construction Y_1 is separable. We must show that $T(t)y \in Y_1$ for all $t > 0$. Suppose that $T_s y \notin Y_1$ for some $s > 0$. Then there is a $\varphi \in Y_1^*$ such that $\varphi(T(s)y) \neq 0$, but $\varphi(T(t)y) = 0$ for each rational $t \geqslant 0$. Since $[Y]$ is reflexive, $i^*(X^*)$, which is weak* dense, is actually norm dense in $[Y]^*$. Hence there is a sequence $\{\psi_n\}$ in X^* with $i^*\psi_n \to \varphi$ in norm in $[Y]^*$. But then

$$\varphi(T(t)y) = \lim_{n\to\infty} (i * \psi_n)(T(t)y),$$

and this limit is uniform on compact subsets of $(0, \infty)$ by the local boundedness of $t \to | T(t)y |$ which follows from Lemma 8. Since the functions $(i^*\psi_n)(T(t)y) = \psi_n(T(t)y)$ are continuous on $(0, \infty)$, the same is true of their uniform limit $\varphi(T(t)y)$. But $\varphi(T(t)y) = 0$ for every rational $t > 0$, hence for every $t > 0$, contradicting $\varphi(T(s)y) \neq 0$. □

We can now finish the proof of Theorem 6. Suppose that (ii) holds. Let $x \in X$ and $\epsilon > 0$ be given. Since T is essential on X we can find $u \in X$ and $t_0 > 0$ with $\| x - T(t_0)u \| < \epsilon/3$. Since Y is dense in X and $T(t_0)$ is bounded we can find $y \in Y$ with $\| T(t_0)u - T(t_0)y \| < \epsilon/3$. By Lemma 9 the closure in $[Y]$ of the span of $\{T(t)y : t > 0\}$ is separable. Call this space Y_1, and let X_1 be the closure of Y_1 in the norm of X. Note that X_1 is a closed invariant subspace for T, and that T is essential on X_1 by construction. Hence by case (i) of our theorem, applied to X_1 and Y_1, there is a $z \in \mathscr{D}^\infty(A) \cap Y_1 \subset \mathscr{D}^\infty(A) \cap Y$ with $\| z - T(t_0)y \| < \epsilon/3$. Hence $\| z - x \| < \epsilon$ and the proof is complete. □

A final remark: Lemma 9 is not true if the hypothesis that $[Y]$ be reflexive is dropped. To see this let X and T be as in Proposition 1, and let Y consist of all essentially bounded functions $f \in X$ with $| f | = \| f \|_p + \| f \|_\infty$. Let $y \in Y$ be the characteristic function of the interval $[0, 1]$. Then any finite linear combination of rational translates of y will be a step function whose intervals of constancy all have rational end points. The L^∞ distance from any such function to an irrational translate of y will be at least $1/2$. Hence no irrational translate of y belongs to Y_1, the closure of the span of $\{T(t)y : t \geqslant 0, t \text{ rational}\}$.

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